

Gromov-Witten theory and Donaldson-Thomas theory, I

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1 May 2004

1 Introduction

1.1 Overview

Let X be a nonsingular, projective, Calabi-Yau 3-fold. Gromov-Witten theory concerns counts of maps of curves to X . The counts are defined in terms of a canonical 0-dimensional virtual fundamental class on the moduli space of maps. The discrete invariants of a map are the genus g of the domain and the degree $\beta \in H_2(X, \mathbb{Z})$ of the image. For every g and β , the Gromov-Witten invariant is the virtual number of genus g , degree β maps. We sum the contributions of all genera with weight u^{2g-2} , where u is a parameter.

The Gromov-Witten invariants have long been expected to be expressible in terms of appropriate curve counts in the target X . A curve in X corresponds to an ideal sheaf on X . The discrete invariants of the ideal sheaf are the holomorphic Euler characteristic χ and the fundamental class $\beta \in H_2(X, \mathbb{Z})$ of the associated curve. Donaldson and Thomas have constructed a canonical 0-dimensional virtual fundamental class on the moduli space of ideal sheaves on X . For every χ and β , the Donaldson-Thomas invariant is the virtual number of the corresponding ideal sheaves. We sum the contributions over χ with weight q^χ , where q is a parameter.

We present here a precise mathematical conjecture relating the Gromov-Witten and Donaldson-Thomas theories of X . Our conjecture is motivated

by the description of Gromov-Witten theory via crystals in [34]. A connection between Gromov-Witten theory and integration over the moduli space of ideal sheaves is strongly suggested there. A related physical conjecture is formulated in [18].

Conjecture. *The change of variables, $e^{iu} = -q$, equates the Gromov-Witten and Donaldson-Thomas theories of X .*

The moduli of maps and sheaves have been related previously by the Gopakumar-Vafa conjecture equating Gromov-Witten invariants to BPS state counts determined by the *cohomology* of the moduli of sheaves [13, 14]. The Gopakumar-Vafa conjecture has been verified in several cases and has been a significant source of motivation. However, there have been difficulties on the mathematical side in selecting an appropriate cohomology theory for the singular moduli of sheaves which arise, see [17].

Donaldson-Thomas theory concerns *integration* over the moduli of sheaves. The subject was defined, along with a construction of the virtual class, by Donaldson and Thomas in [10, 40] with motivation from several sources, see [1, 3, 41]. As the Donaldson-Thomas invariant is similar to the Euler characteristic of the moduli of sheaves, a philosophical connection between Gromov-Witten invariants and the cohomology of the moduli of sheaves is implicit in our work. However, the Donaldson-Thomas invariant is *not* the Euler characteristic.

As evidence for our conjecture, we present a proof in the toric local Calabi-Yau case via the virtual localization formula for Donaldson-Thomas theory. The proof depends upon partial evaluations of the topological vertex on the Gromov-Witten side.

1.2 General 3-folds

We believe the Gromov-Witten/Donaldson-Thomas correspondence holds for all 3-folds. Donaldson-Thomas theory has a natural supply of observables constructed from the Chern classes of universal sheaves. These Chern classes should correspond to insertions in Gromov-Witten theory, see [23]. The degree 0 case, where no insertions are required, is discussed in Section 2 below. For primary fields, a complete GW/DT correspondence for all 3-folds is conjectured in [25]. For descendent fields, the correspondence is precisely formulated for the descendants of a point in [25].

We conjecture the *equivariant vertex* [30], discussed briefly in Section 4.9 below, has the same relation to general cubic Hodge integrals as the topological vertex does to Calabi-Yau Hodge integrals [9]. A closely related issue is the precise formulation of the GW/DT correspondence for *all* descendent fields. We will pursue the topic in a future paper.

1.3 GW theory

Gromov-Witten theory is defined via integration over the moduli space of stable maps. Let X be a nonsingular, projective, Calabi-Yau 3-fold. Let $\overline{M}_g(X, \beta)$ denote the moduli space of stable maps from connected genus g curves to X representing the class $\beta \in H_2(X, \mathbb{Z})$, and let

$$N_{g,\beta} = \int_{[\overline{M}_g(X, \beta)]^{vir}} 1,$$

denote the corresponding Gromov-Witten invariant. Foundational aspects of the theory are treated, for example, in [4, 5, 20].

Let $F'_{GW}(X; u, v)$ denote the reduced Gromov-Witten potential of X ,

$$F'_{GW}(X; u, v) = \sum_{\beta \neq 0} \sum_{g \geq 0} N_{g,\beta} u^{2g-2} v^\beta,$$

omitting the constant maps. The reduced partition function,

$$Z'_{GW}(X; u, v) = \exp F'_{GW}(X; u, v),$$

generates disconnected Gromov-Witten invariants of X excluding constant contributions.

Let $Z'_{GW}(X; u)_\beta$ denote the reduced partition function of degree β invariants,

$$Z'_{GW}(X; u, v) = 1 + \sum_{\beta \neq 0} Z'_{GW}(X; u)_\beta v^\beta.$$

1.4 DT theory

Donaldson-Thomas theory is defined via integration over the moduli space of ideal sheaves. Let X be a nonsingular, projective, Calabi-Yau 3-fold. An

ideal sheaf is a torsion-free sheaf of rank 1 with trivial determinant. Each ideal sheaf \mathcal{I} injects into its double dual,

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{I}^{\vee\vee}.$$

As $\mathcal{I}^{\vee\vee}$ is reflexive of rank 1 with trivial determinant,

$$\mathcal{I}^{\vee\vee} \cong \mathcal{O}_X,$$

see [32]. Each ideal sheaf \mathcal{I} determines a subscheme $Y \subset X$,

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0.$$

The maximal dimensional components of Y (weighted by their intrinsic multiplicities) determine an element,

$$[Y] \in H_*(X, \mathbb{Z}).$$

Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves \mathcal{I} satisfying

$$\chi(\mathcal{O}_Y) = n,$$

and

$$[Y] = \beta \in H_2(X, \mathbb{Z}).$$

Here, χ denotes the holomorphic Euler characteristic. The moduli space $I_n(X, \beta)$ is isomorphic to the Hilbert scheme of curves of X [26].

The Donaldson-Thomas invariant is defined via integration against the dimension 0 virtual class,

$$\tilde{N}_{n, \beta} = \int_{[I_n(X, \beta)]^{vir}} 1.$$

Foundational aspects of the theory are treated in [26, 40].

Let $Z_{DT}(X; q, v)$ be the partition function of the Donaldson-Thomas theory of X ,

$$Z_{DT}(X; q, v) = \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{n \in \mathbb{Z}} \tilde{N}_{n, \beta} q^n v^\beta.$$

An elementary verification shows, for fixed β , the invariant $\tilde{N}_{n, \beta}$ vanishes for sufficiently negative n since the corresponding moduli spaces of ideal sheaves are empty.

The degree 0 moduli space $I_n(X, 0)$ is isomorphic to the Hilbert scheme of n points on X . The degree 0 partition function,

$$Z_{DT}(X; q)_0 = \sum_{n \geq 0} \tilde{N}_{n,\beta} q^n,$$

plays a special role in the theory. The MacMahon function,

$$M(q) = \prod_{n \geq 1} \frac{1}{(1 - q^n)^n},$$

is the generating series for 3-dimensional partitions, see [37].

Conjecture 1. The degree 0 partition function is determined by

$$Z_{DT}(X; q)_0 = M(-q)^{\chi(X)},$$

where $\chi(X)$ is the topological Euler characteristic.

The reduced partition function $Z'_{DT}(X; q, v)$ is defined by quotienting by the degree 0 function,

$$Z'_{DT}(X; q, v) = Z_{DT}(X; q, v) / Z_{DT}(X; q)_0.$$

Let $Z'_{DT}(X; q)_\beta$ denote the reduced partition function of degree $\beta \neq 0$ invariants,

$$Z'_{DT}(X; q, v) = 1 + \sum_{\beta \neq 0} Z'_{DT}(X; q)_\beta v^\beta.$$

Conjecture 2. The reduced series $Z'_{DT}(X; q)_\beta$ is a rational function of q symmetric under the transformation $q \mapsto 1/q$.

We now state our main conjecture relating the Gromov-Witten theory and the Donaldson-Thomas theory of a Calabi-Yau 3-fold X .

Conjecture 3. The change of variables $e^{iu} = -q$ equates the reduced partition functions:

$$Z'_{GW}(X; u, v) = Z'_{DT}(X; -e^{iu}, v).$$

The change of variables in Conjecture 3 is well-defined by Conjecture 2. Gromov-Witten and Donaldson-Thomas theory may be viewed as expansions

of a single partition functions at different points. Conjecture 3 can be checked order by order in u and q only if an effective bound on the degree of the rational function in Conjecture 2 is known.

1.5 Integrality

The Gopakumar-Vafa conjecture for Calabi-Yau 3-folds predicts the following form for the reduced Gromov-Witten partition function via the change of variables $e^{iu} = -q$,

$$Z'_{GW}(X; u)_\beta = q^r \frac{f(q)}{\prod_{i=1}^k (1 - (-q)^{s_i})^2}, \quad f \in \mathbb{Z}[q], \quad r \in \mathbb{Z}, \quad s_i > 0.$$

In particular, by the Gopakumar-Vafa conjecture, $Z'_{GW}(X; u)_\beta$ defines a series in q with *integer* coefficients.

Conjecture 3 identifies the q series with the reduced partition function of the Donaldson-Thomas theory of X . Integrality of the Donaldson-Thomas invariants holds by construction (as no orbifolds occur). We may refine Conjecture 2 above to fit the form of the Gopakumar-Vafa conjecture.

1.6 Gauge/string dualities

In spirit, our conjecture is similar to a gauge/string duality with Donaldson-Thomas theory on the gauge side and Gromov-Witten theory on the string side. While at present we are not aware of a purely gauge theoretic interpretation of Donaldson-Thomas theory, there are various indications such an interpretation should exist. Most importantly, the equivariant vertex measure which appears in the equivariant localization formula for Donaldson-Thomas theory, see Section 4.9, is identical to the equivariant vertex in noncommutative Yang-Mills theory [29]. We plan to investigate the issue further.

The interplay between gauge fields and strings is one of the central themes in modern theoretical and mathematical physics [36]. In particular, the conjectural Chern-Simons/string duality of Gopakumar and Vafa [12] was a source of many insights into the Gromov-Witten theory of Calabi-Yau 3-folds. As a culmination of these developments, the *topological vertex* was introduced in [2]. The topological vertex is a certain explicit function of three partitions λ, μ, ν , and the genus expansion parameter u , which is an elementary building block for constructing the Gromov-Witten invariants of

arbitrary local toric Calabi-Yau 3-folds. The gauge/string duality seems to hold in a broader context, see [23], [35] for evidence in the Fano case.

In [34], the topological vertex was interpreted as counting 3-dimensional partitions π with asymptotics λ, μ, ν along the coordinate axes. The variable $q = e^{iu}$ couples to the volume of the partition π in the enumeration. The global data obtained by gluing such 3-dimensional partitions according to the gluing rules of the topological vertex was observed in [18, 34] to naturally corresponds to torus invariant ideal sheaves in the target 3-fold X . The main mathematical result of our paper is the identification of the topological vertex expansion with the equivariant localization formula for the Donaldson-Thomas theory of the local Calabi-Yau geometry.

The GW/DT correspondence is conjectured to hold for *all* Calabi-Yau 3-folds. While several motivations for the correspondence came from local Calabi-Yau geometry, new methods of attack will be required to study the full GW/DT correspondence.

A relation between Gromov-Witten theory and gauge theory on the same space X has been observed previously in four (real) dimensions in the context of Seiberg-Witten invariants [39]. There, a deformation of the Seiberg-Witten equations by a 2-form yields solutions concentrated near the zero locus of the 2-form, an embedded curve.

We expect, in our case, the sheaf-theoretic description of curves will be identified with a deformed version of solutions to some gauge theory problem. An outcome should be a natural method of deriving the Donaldson-Thomas measure. The gauge theory in question is a deformation of the twisted maximally supersymmetric Yang-Mills theory compactified on our 3-fold X [29]. The theory, discussed in [3], has BPS solutions and generalized instantons. The expansion of the super-Yang-Mills action about these solutions gives rise to a quadratic form with bosonic and fermionic determinants which should furnish the required measure [27, 28].

In case $X = \mathbf{C}^3$, the deformation in question is the passage to the non-commutative \mathbb{R}^6 , see [31, 42]. Ordinary gauge theories have typically non-compact moduli spaces of BPS solutions. It is customary in mathematics to compactify these spaces by replacing holomorphic bundles by coherent sheaves. The physical consequences of such a replacement are usually quite interesting and lead to many insights [16, 19, 24]. Sometimes the “compactified” space is non-empty while the original space is empty. Our problem corresponds to $U(1)$ gauge fields which do not support nontrivial instantons, while the compactified moduli space of instantons is non-empty and coincides

with the Hilbert scheme of curves of given topology on X .

1.7 Acknowledgments

We thank J. Bryan, A. Iqbal, M. Kontsevich, Y. Soibelman, R. Thomas, and C. Vafa for related discussions.

D. M. was partially supported by a Princeton Centennial graduate fellowship. A. O. was partially supported by DMS-0096246 and fellowships from the Sloan and Packard foundations. R. P. was partially supported by DMS-0071473 and fellowships from the Sloan and Packard foundations.

2 Degree 0

2.1 GW theory

Let X be a nonsingular, projective 3-fold (not necessarily Calabi-Yau). The degree 0 potential $F_{GW}(X; u)_0$ may be separated as:

$$F_{GW}(X; u)_0 = F_{X,0}^0 + F_{X,0}^1 + \sum_{g \geq 2} F_{X,0}^g.$$

The genus 0 and 1 contributions in degree 0 are not constants, the variables of the classical cohomology appear explicitly. Formulas can be found, for example, in [35].

We will be concerned here with the higher genus terms. For $g \geq 2$, a virtual class calculation yields,

$$F_{X,0}^g = (-1)^g \frac{u^{2g-2}}{2} \int_X (c_3(X) - c_1(X)c_2(X)) \cdot \int_{\overline{M}_g} \lambda_{g-1}^3,$$

where c_i and λ_i denote the Chern classes of the tangent bundle T_X and the Hodge bundle \mathbb{E}_g respectively. Define the degree 0 partition function of Gromov-Witten theory by

$$Z_{GW}(X; u)_0 = \exp \left(\sum_{g \geq 2} F_{X,0}^g \right).$$

The Hodge integrals which arise have been computed in [11],

$$\int_{\overline{M}_g} \lambda_{g-1}^3 = \frac{|B_{2g}|}{2g} \frac{|B_{2g-2}|}{2g-2} \frac{1}{(2g-2)!}, \quad (1)$$

where B_{2g} and B_{2g-2} are Bernoulli numbers.

Using the Euler-Maclaurin formula, the asymptotic relation,

$$Z_{GW}(X; u)_0 \sim M(e^{iu})^{\frac{1}{2} \int_X c_3(X) - c_1(X)c_2(X)}, \quad (2)$$

may be derived from (1). The precise meaning of (2) is the following: the logarithms of both sides have identical $o(1)$ -tails in their $u \rightarrow 0$ asymptotic expansion.

2.2 DT theory

We now turn to the degree 0 partition function for the Donaldson-Thomas theory of X . The first issue is the construction of the virtual class in Donaldson-Thomas theory.

In [40], the Donaldson-Thomas theory of X is defined only in the Calabi-Yau and Fano cases. In fact, the arguments of [40] use only the existence of an anticanonical section on X .

Lemma 1. *Let X be a nonsingular, projective 3-fold with*

$$H^0(X, \wedge^3 T_X) \neq 0,$$

then $I_n(X, \beta)$ carries a canonical perfect obstruction theory.

Under the hypotheses of Lemma 1, the Donaldson-Thomas theory of X is constructed for *higher* rank sheaves as well as the rank 1 case of ideal sheaves. The connection, if any, between Gromov-Witten theory and the higher rank Donaldson-Thomas theories is not clear to us.

The technical condition required in [40] for the construction of the perfect obstruction theory and the virtual class $[I_n(X, \beta)]^{vir}$ is the vanishing of traceless $\text{Ext}_0^3(\mathcal{I}, \mathcal{I})$ for all $[\mathcal{I}] \in I_n(X, \beta)$. In fact, $\text{Ext}_0^3(\mathcal{I}, \mathcal{I})$ vanishes for every ideal sheaf on a nonsingular, projective 3-fold X [26]. Hence, Donaldson-Thomas theory is well-defined in rank 1 for *all* X .

For simplicity, let us assume the vanishing of the higher cohomology of the structure sheaf,

$$H^i(X, \mathcal{O}_X) = 0, \quad (3)$$

for $i \geq 1$. Then, $\text{Ext}_0(\mathcal{I}, \mathcal{I})$ equals $\text{Ext}(\mathcal{I}, \mathcal{I})$.

Lemma 2. *Let X be a nonsingular, projective, 3-fold satisfying (3). Then,*

$$\mathrm{Ext}^3(\mathcal{I}, \mathcal{I}) = 0,$$

for all $[\mathcal{I}] \in I_n(X, 0)$.

Proof. By Serre duality for Ext ,

$$\mathrm{Ext}^3(\mathcal{I}, \mathcal{I}) = \mathrm{Ext}^0(\mathcal{I}, \mathcal{I} \otimes K_X)^\vee,$$

where K_X denotes the canonical bundle. We must therefore prove

$$\mathrm{Hom}(\mathcal{I}, \mathcal{I} \otimes K_X) = 0.$$

Let $U \subset X$ be the complement of the support of Y . Since \mathcal{I} restricts to \mathcal{O}_U on U ,

$$\mathrm{Hom}(\mathcal{I}|_U, \mathcal{I}|_U \otimes K_U) = \Gamma(U, K_U) = H^0(X, K_X).$$

The last equality is obtained from the extension of sections since Y has at most 1-dimensional support. Since \mathcal{I} is torsion-free, the restriction,

$$\mathrm{Hom}(\mathcal{I}, \mathcal{I} \otimes K_X) \rightarrow \mathrm{Hom}(\mathcal{I}|_U, \mathcal{I}|_U \otimes K_U),$$

is injective. Since $h^0(X, K_X) = h^3(X, \mathcal{O}_X)$, the Lemma is proven. \square

The proof of the vanishing of $\mathrm{Ext}_0^3(\mathcal{I}, \mathcal{I})$ in the presence of higher cohomology of the structure sheaf is similar, see [26].

The virtual dimension of $I_n(X, 0)$ is 0 for general 3-folds X . A simple calculation from the definitions yields the following result.

Lemma 3. $\tilde{N}_{1,0} = - \int_X c_3(X) - c_1(X)c_2(X)$.

Proof. The moduli space $I_1(X, 0)$ is the nonsingular 3-fold X . The tangent bundle is $\mathrm{Ext}_0^1(\mathcal{I}, \mathcal{I})$, and the obstruction bundle is $\mathrm{Ext}_0^2(\mathcal{I}, \mathcal{I})$. Using Serre duality and the local-to-global spectral sequence for Ext , we find the obstruction bundle is isomorphic to $(T_X \otimes K_X)^\vee$. Then,

$$\tilde{N}_{1,0} = - \int_X c_3(T_X \otimes K_X) = - \int_X c_3(X) - c_1(X)c_2(X),$$

completing the proof. \square

The degree 0 Gromov-Witten and Donaldson-Thomas theories are already related by Lemma 3. However, we make a stronger connection generalizing Conjecture 1.

Conjecture 1'. The degree 0 Donaldson-Thomas partition function for a 3-fold X is determined by:

$$Z_{DT}(X; q)_0 = M(-q)^{f_X c_3(T_X \otimes K_X)}.$$

We will present a proof of Conjecture 1' in case X is a nonsingular toric 3-fold in [25].

The series $M(q)$ arises naturally in the computation of the Euler characteristic of the Hilbert scheme of points of a 3-fold [8]. It would be interesting to find a direct connection between the degree 0 Donaldson-Thomas invariants and the Euler characteristics of $I_n(X, 0)$ in the Calabi-Yau case.

3 Local Calabi-Yau geometry

3.1 GW theory

Let S be a nonsingular, projective, toric, Fano surface with canonical bundle K_S . The Gromov-Witten theory of the local Calabi-Yau geometry of S is defined via an excess integral. Denote the universal curve and universal map over the moduli space of stable maps to S by:

$$\pi : U \rightarrow \overline{M}_g(S, \beta),$$

$$\mu : U \rightarrow S.$$

Then,

$$N_{g, \beta} = \int_{[\overline{M}_g(S, \beta)]^{vir}} e(R^1 \pi_* \mu^* K_S),$$

for $0 \neq \beta \in H_2(S, \mathbb{Z})$. The reduced partition function $Z'_{GW}(X; u, v)$ is defined in terms of the local invariants $N_{g, \beta}$ as before.

3.2 DT theory

Let X be the projective bundle $\mathbf{P}(K_S \oplus \mathcal{O}_S)$ over the surface S . The Donaldson-Thomas theory of X is well-defined in every rank by the following observation.

Lemma 4. *X has an anticanonical section.*

Proof. Consider the fibration $\pi : X \rightarrow S$. We have,

$$\wedge^3 T_X = T_\pi \otimes \pi^*(\wedge^2 T_S),$$

where T_π is the π -vertical tangent line.

Let V denote the vector bundle $K_S \oplus \mathcal{O}_S$ on S . The π -relative Euler sequence is:

$$0 \rightarrow \mathcal{O}_X \rightarrow \pi^*(V) \otimes \mathcal{O}_{\mathbf{P}(V)}(1) \rightarrow T_\pi \rightarrow 0.$$

Hence,

$$T_\pi = \wedge^2 \pi^*(V) \otimes \mathcal{O}_{\mathbf{P}(V)}(2).$$

we conclude,

$$\wedge^3 T_X = \wedge^2 \pi^*(V) \otimes \mathcal{O}_{\mathbf{P}(V)}(2) \otimes \pi^*(\wedge^2 T_S) = \mathcal{O}_{\mathbf{P}(V)}(2).$$

However, since

$$H^0(X, \mathcal{O}_{\mathbf{P}(V)}(2)) = H^0(S, \text{Sym}^2 V^*) \neq 0,$$

the Lemma is proven. \square

For classes $\beta \in H_2(S, \mathbb{Z})$, we define the reduced partition function for the Donaldson-Thomas theory of the Calabi-Yau geometry of S by

$$\mathsf{Z}'_{DT}(S; q)_\beta = \mathsf{Z}'_{DT}(X; q)_\beta.$$

While X is not Calabi-Yau, the Donaldson-Thomas theory of X is still well-defined by Lemma 1.

We will prove Conjectures 2 and 3 are true for the local Calabi-Yau geometry of toric Fano surfaces by virtual localization.

3.3 Local curves

The constructions above also define the local Calabi-Yau theory of the curve \mathbf{P}^1 with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The proof of Conjectures 2 and 3 for local surfaces given in Section 4 below is valid for the local \mathbf{P}^1 case.

The Gromov-Witten theory of a local Calabi-Yau curve of arbitrary genus has been defined in [6]. We believe the GW/DT correspondence holds for these geometries as well [7].

4 Localization in Donaldson-Thomas theory

4.1 Toric geometry

Let X be a nonsingular, projective, toric 3-fold. Let \mathbf{T} be the 3-dimensional complex torus acting on X . Let $\Delta(X)$ denote the Newton polyhedron of X determined by a polarization. The polyhedron $\Delta(X)$ is the image of X under the moment map.

The vertices of $\Delta(X)$ correspond to fixed points $X^\mathbf{T} = \{X_\alpha\}$ of the \mathbf{T} -action. For each X_α , there is a canonical, \mathbf{T} -invariant, affine open chart,

$$U_\alpha \cong \mathbf{A}^3,$$

centered at X_α . We may choose coordinates t_i on \mathbf{T} and coordinates x_i on U_α for which the \mathbf{T} -action on U_α is determined by

$$(t_1, t_2, t_3) \cdot x_i = t_i x_i. \quad (4)$$

In these coordinates, the tangent representation X_α has character

$$t_1^{-1} + t_2^{-1} + t_3^{-1}.$$

We will use the covering $\{U_\alpha\}$ of X to compute Čech cohomology.

The \mathbf{T} -invariant lines of X correspond to the edges of $\Delta(X)$. More precisely, if

$$C_{\alpha\beta} \subset X$$

is a \mathbf{T} -invariant line incident to the fixed points X_α and X_β , then $C_{\alpha\beta}$ corresponds to an edge of $\Delta(X)$ joining the vertices X_α and X_β .

The geometry of $\Delta(X)$ near the edge is determined by the normal bundle $\mathcal{N}_{C_{\alpha\beta}/X}$. If

$$\mathcal{N}_{C_{\alpha\beta}/X} = \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta})$$

then the transition functions between the charts U_α and U_β can be taken in the form

$$(x_1, x_2, x_3) \mapsto (x_1^{-1}, x_2 x_1^{-m_{\alpha\beta}}, x_3 x_1^{-m'_{\alpha\beta}}). \quad (5)$$

The curve $C_{\alpha\beta}$ is defined in these coordinates by $x_2 = x_3 = 0$.

4.2 Moduli of ideal sheaves

The \mathbf{T} -action on X canonically lifts to the moduli space of ideal sheaves $I_n(X, \beta)$. The perfect obstruction theory constructed by Thomas is canonically \mathbf{T} -equivariant [26, 40]. The virtual localization formula reduces integration against $[I_n(X, \beta)]^{vir}$ to a sum fixed point contributions [15].

The first step is to determine the \mathbf{T} -fixed points of $I_n(X, \beta)$. If

$$[\mathcal{I}] \in I_n(X, \beta)$$

is \mathbf{T} -fixed, then the associated subscheme $Y \subset X$ must be preserved by the torus action. Hence, Y must be supported on the \mathbf{T} -fixed points X_α and the \mathbf{T} -invariant lines connecting them.

Since \mathcal{I} is \mathbf{T} -fixed on each open set, \mathcal{I} must be defined on U_α by a monomial ideal,

$$I_\alpha = \mathcal{I}|_{U_\alpha} \subset \mathbf{C}[x_1, x_2, x_3],$$

and may also be viewed as a 3-dimensional partition π_α

$$\pi_\alpha = \left\{ (k_1, k_2, k_3), \prod_1^3 x_i^{k_i} \notin I_\alpha \right\} \subset \mathbb{Z}_{\geq 0}^3.$$

The ideals I_α are 1-dimensional. The corresponding partitions π_α may be infinite in the direction of the coordinate axes.

The asymptotics of π_α in the coordinate directions are described by three ordinary 2-dimensional partitions. In particular, in the direction of the \mathbf{T} -invariant curve $C_{\alpha\beta}$, we have the partition $\lambda_{\alpha\beta}$ with the following diagram:

$$\begin{aligned} \lambda_{\alpha\beta} &= \left\{ (k_2, k_3), \forall k_1 x_1^{k_1} x_2^{k_2} x_3^{k_3} \notin I_\alpha \right\} \\ &= \left\{ (k_2, k_3), x_2^{k_2} x_3^{k_3} \notin I_{\alpha\beta} \right\}, \end{aligned}$$

where

$$I_{\alpha\beta} = \mathcal{I}|_{U_\alpha \cap U_\beta} \subset \mathbf{C}[x_1^{\pm 1}, x_2, x_3].$$

In summary, a \mathbf{T} -fixed ideal sheaf \mathcal{I} can be described in terms of the following data:

- (i) a 2-dimensional partition $\lambda_{\alpha\beta}$ assigned to each edge of $\Delta(X)$,
- (ii) a 3-dimensional partition π_α assigned to each vertex of $\Delta(X)$ such that the asymptotics of π_α in the three coordinate directions is given by the partitions $\lambda_{\alpha\beta}$ assigned to the corresponding edges.

4.3 Melting crystal interpretation

The partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}$ corresponding to a \mathbf{T} -fixed ideal sheaf \mathcal{I} can be given a melting crystal interpretation [34]. Consider the weights of the \mathbf{T} -action on

$$H^0(X, \mathcal{O}_Y(d)).$$

For large d , the corresponding points of \mathbb{Z}^3 can be described as follows.

Scale the Newton polyhedron $\Delta(X)$ by a factor of d . Near each corner of $d\Delta(X)$, the intersection $\mathbb{Z}^3 \cap d\Delta(X)$ looks like a standard $\mathbb{Z}_{\geq 0}^3$, so we can place the corresponding partition π_α there. Since d is large and since, by construction π_α and π_β agree along the edge joining them, a global combinatorial object emerges.

One can imagine the points of $\mathbb{Z}^3 \cap d\Delta(X)$ are atoms in a crystal and, as the crystal is melting or dissolving, some of the atoms near the corners and along the edges are missing. These missing atoms are described by the partitions π_α and $\lambda_{\alpha\beta}$. They are precisely the weights of the \mathbf{T} -action on $H^0(X, \mathcal{O}_Y(d))$.

4.4 Degree and Euler characteristic

Let $[\mathcal{I}] \in I_n(X, \beta)$ be a \mathbf{T} -fixed ideal sheaf on X described by the partition data $\{\pi_\alpha, \lambda_{\alpha\alpha'}\}$. We see

$$\beta = \sum_{\alpha, \alpha'} |\lambda_{\alpha\alpha'}| [C_{\alpha\alpha'}],$$

where $|\lambda|$ denotes the size of a partition λ , the number of squares in the diagram.

For 3-dimensional partitions π one can similarly define their size $|\pi|$ by the number of cubes in their diagram. Since the partitions π_α may be infinite along the coordinate axes, the number $|\pi_\alpha|$ so defined will often be infinite.

We define the renormalized volume $|\pi_\alpha|$ as follows. Let $\lambda_{\alpha\beta_i}$, $i = 1, 2, 3$, be the asymptotics of π_α . We set

$$|\pi_\alpha| = \# \{ \pi_\alpha \cap [0, \dots, N]^3 \} - (N+1) \sum_1^3 |\lambda_{\alpha\beta_i}|, \quad N \gg 0.$$

The renormalized volume is independent of the cut-off N as long as N is sufficiently large. The number $|\pi_\alpha|$ so defined may be negative.

Given $m, m' \in \mathbb{Z}$ and a partition λ , we define

$$f_{m,m'}(\lambda) = \sum_{(i,j) \in \lambda} [m(i-1) + m'(j-1) + 1].$$

Each edge of $\Delta(X)$ is assigned a pair of integers $(m_{\alpha\beta}, m'_{\alpha\beta})$ from the normal bundle of the associated \mathbf{T} -invariant line and a partition $\lambda_{\alpha\beta}$ from the \mathbf{T} -fixed ideal sheaf \mathcal{I} . By definition, we set

$$f(\alpha, \beta) = f_{m_{\alpha\beta}, m'_{\alpha\beta}}(\lambda_{\alpha\beta}). \quad (6)$$

Lemma 5. $\chi(\mathcal{O}_Y) = \sum_\alpha |\pi_\alpha| + \sum_{\alpha, \beta} f(\alpha, \beta)$.

Proof. The result is an elementary calculation in toric geometry. For example, a computation of $\chi(\mathcal{O}_Y)$ using the Čech cover defined by $\{U_\alpha\}$ immediately yields the result. \square

4.5 The \mathbf{T} -fixed obstruction theory

The moduli space $I_n(X, \beta)$ carries a \mathbf{T} -equivariant perfect obstruction theory,

$$E_0 \rightarrow E_1,$$

see [26, 40]. Assume the virtual dimension of $I_n(X, \beta)$ is 0. The virtual localization formula [15] may be stated as follows,

$$\int_{[I_n(X, \beta)]^{vir}} 1 = \sum_{[\mathcal{I}] \in I_n(X, \beta)^{\mathbf{T}}} \int_{[S(\mathcal{I})]^{vir}} \frac{e(E_1^m)}{e(E_0^m)}.$$

Here, $S(\mathcal{I})$ denotes the \mathbf{T} -fixed subscheme of $I_n(X, \beta)$ supported at the point $[\mathcal{I}]$, and E_0^m , E_1^m denote the nonzero \mathbf{T} -weight spaces. The virtual class, $[S(\mathcal{I})]^{vir}$, is determined by the \mathbf{T} -fixed obstruction theory.

We first prove $S(\mathcal{I})$ is the reduced point $[\mathcal{I}]$. It suffices to prove the Zariski tangent space to $I_n(X, \beta)$ at $[\mathcal{I}]$ contains no trivial subrepresentations. Since X is toric, all the higher cohomologies of \mathcal{O}_X vanish,

$$H^i(X, \mathcal{O}_X) = 0,$$

for $i \geq 0$. Hence, the traceless condition is satisfied, and the Zariski tangent space of $I_n(X, \beta)$ at $[\mathcal{I}]$ is $\text{Ext}^1(\mathcal{I}, \mathcal{I})$.

Lemma 6. *Let $[\mathcal{I}] \in I_n(X, \beta)$ be a \mathbf{T} -fixed point. The \mathbf{T} -representation,*

$$\text{Ext}^1(\mathcal{I}, \mathcal{I}),$$

contains no trivial subrepresentations.

Proof. From the \mathbf{T} -equivariant ideal sheaf sequence,

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0, \quad (7)$$

we obtain a sequence of \mathbf{T} -representations,

$$\rightarrow \text{Ext}^0(\mathcal{I}, \mathcal{O}_Y) \rightarrow \text{Ext}^1(\mathcal{I}, \mathcal{I}) \rightarrow \text{Ext}^1(\mathcal{I}, \mathcal{O}_X) \rightarrow .$$

The left term, $\text{Ext}^0(\mathcal{I}, \mathcal{O}_Y)$, does not contain trivial representations by Lemma 7 below.

We will prove the right term, $\text{Ext}^1(\mathcal{I}, \mathcal{O}_X)$, also does not contain trivial representations. By Serre duality, it suffices to study the representation,

$$\text{Ext}^2(\mathcal{O}_X, \mathcal{I} \otimes K_X) = H^2(X, \mathcal{I} \otimes K_X).$$

The long exact sequence in cohomology obtained from (7) by tensoring with K_X and the vanishings,

$$H^1(X, K_X) = H^2(X, K_X) = 0,$$

together yield a \mathbf{T} -equivariant isomorphism,

$$H^1(X, \mathcal{O}_Y \otimes K_X) \xrightarrow{\sim} H^2(X, \mathcal{I} \otimes K_X).$$

The first Cech cohomology of $\mathcal{O}_Y \otimes K_X$ is computed via the representations

$$H^0(U_{\alpha\beta}, \mathcal{O}_Y \otimes K_X),$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Here we use the Cech cover defined in Section 4.1. An elementary argument shows these representations contain no trivial subrepresentations. \square

Lemma 7. $\text{Ext}^k(\mathcal{I}, \mathcal{O}_Y)$ contains no trivial subrepresentations.

Proof. By the local-to-global spectral sequence, it suffices to prove

$$H^i(\mathcal{E}xt^j(\mathcal{I}, \mathcal{O}_Y))$$

contains no trivial subrepresentations for all i and j . By a Čech cohomology calculation, it then suffices to prove

$$H^0(U_\alpha, \mathcal{E}xt^j(\mathcal{I}_\alpha, \mathcal{O}_{Y_\alpha})), H^0(U_{\alpha\beta}, \mathcal{E}xt^j(\mathcal{I}_{\alpha\beta}, \mathcal{O}_{Y_{\alpha\beta}}))$$

contain no trivial subrepresentations. Triple intersections need not be considered since $\mathcal{O}_{Y_{\alpha\beta\gamma}}$ vanishes.

We will study $\mathcal{E}xt^j(\mathcal{I}_\alpha, \mathcal{O}_{Y_\alpha})$ on U_α via the \mathbf{T} -equivariant Taylor resolution of the monomial ideal \mathcal{I}_α . The argument for $\mathcal{E}xt^j(\mathcal{I}_{\alpha\beta}, \mathcal{O}_{Y_{\alpha\beta}})$ on $U_{\alpha\beta}$ is identical.

Let \mathcal{I}_α be generated by the monomials m_1, \dots, m_s . For each subset

$$T \subset \{1, \dots, s\},$$

let

$$m_T = x^{r(T)} = \text{least common multiple of } \{m_i \mid i \in T\}.$$

For $1 \leq t \leq s$, let F_t be the free $\Gamma(U_\alpha)$ -module with basis e_T indexed by subsets $T \subset \{1, \dots, s\}$ of size t .

A differential $d : F_t \rightarrow F_{t-1}$ is defined as follows. Given a subset T of size t , let $T = \{i_1, \dots, i_t\}$ where $i_1 < \dots < i_t$. Let

$$d(e_T) = \sum_{T' = T \setminus \{i_k\}} (-1)^k x^{r_T - r_{T'}}.$$

The Taylor resolution,

$$0 \rightarrow F_s \rightarrow \dots \rightarrow F_2 \rightarrow F_1 \rightarrow \mathcal{I}_\alpha \rightarrow 0,$$

is exact [38]. Moreover, the resolution is equivariant with \mathbf{T} -weight $r(T)$ on the generator $e(T)$.

The weights of the generators of F_t are weights of monomials in \mathcal{I}_α . However, the weights of the \mathbf{T} -representation \mathcal{O}_{Y_α} are precisely not equal to weights of monomials in \mathcal{I}_α . Hence, $\text{Hom}(F_t, \mathcal{O}_{Y_\alpha})$ contains no trivial subrepresentations. We then conclude $\mathcal{E}xt^j(\mathcal{I}_\alpha, \mathcal{O}_Y)$ contains no trivial subrepresentations by computing via the Taylor resolution of \mathcal{I}_α . \square

The obstruction space at $[\mathcal{I}] \in I_n(X, \beta)$ of the perfect obstruction theory is $\mathrm{Ext}^2(\mathcal{I}, \mathcal{I})$. The following Lemma implies the \mathbf{T} -fixed obstruction theory at $[\mathcal{I}]$ is trivial.

Lemma 8. *$\mathrm{Ext}^2(\mathcal{I}, \mathcal{I})$ contains no trivial subrepresentations.*

Proof. From the \mathbf{T} -equivariant ideal sheaf sequence, we obtain,

$$\rightarrow \mathrm{Ext}^1(\mathcal{I}, \mathcal{O}_Y) \rightarrow \mathrm{Ext}^2(\mathcal{I}, \mathcal{I}) \rightarrow \mathrm{Ext}^2(\mathcal{I}, \mathcal{O}_X) \rightarrow .$$

The left term, $\mathrm{Ext}^1(\mathcal{I}, \mathcal{O}_Y)$, does not contain trivial representations by Lemma 7 above.

We will prove the right term, $\mathrm{Ext}^2(\mathcal{I}, \mathcal{O}_X)$, also does not contain trivial representations. By Serre duality, it suffices to study the representation,

$$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{I} \otimes K_X) = H^1(X, \mathcal{I} \otimes K_X).$$

The long exact sequence in cohomology obtained from by tensoring the ideal sheaf sequence with K_X and the vanishings,

$$H^0(X, K_X) = H^1(X, K_X) = 0,$$

together yield a \mathbf{T} -equivariant isomorphism,

$$H^0(X, \mathcal{O}_Y \otimes K_X) \xrightarrow{\sim} H^1(X, \mathcal{I} \otimes K_X).$$

The space of global sections of $\mathcal{O}_Y \otimes K_X$ is computed via the representations

$$H^0(U_\alpha, \mathcal{O}_Y \otimes K_X).$$

As before, an elementary argument shows these representations contain no trivial subrepresentations. \square

The virtual localization formula may then be written as

$$\int_{[I_n(X, \beta)]^{vir}} 1 = \sum_{[\mathcal{I}] \in I_n(X, \beta)^{\mathbf{T}}} \frac{e(\mathrm{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}))}.$$

A calculation of the virtual representation

$$\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}) - \mathrm{Ext}^2(\mathcal{I}, \mathcal{I})$$

is required for the evaluation of the virtual localization formula.

4.6 Virtual tangent space

The virtual tangent space at \mathcal{I} is given by

$$\mathcal{T}_{[\mathcal{I}]} = \text{Ext}^1(\mathcal{I}, \mathcal{I}) - \text{Ext}^2(\mathcal{I}, \mathcal{I}) = \chi(\mathcal{O}, \mathcal{O}) - \chi(\mathcal{I}, \mathcal{I})$$

where

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_{i=0}^3 (-1)^i \text{Ext}^i(\mathcal{F}, \mathcal{G}).$$

We can compute each Euler characteristic using the local-to-global spectral sequence

$$\begin{aligned} \chi(\mathcal{I}, \mathcal{I}) &= \sum_{i,j=0}^3 (-1)^{i+j} H^i(\mathcal{E}xt^j(\mathcal{I}, \mathcal{I})) \\ &= \sum_{i,j=0}^3 (-1)^{i+j} \mathfrak{C}^i(\mathcal{E}xt^j(\mathcal{I}, \mathcal{I})), \end{aligned}$$

where, in the second line, we have replaced the cohomology terms with the Čech complex with respect to the open affine cover $\{U_\alpha\}$. Though these modules are infinite-dimensional, they have finite-dimensional weight spaces and, therefore, their \mathbf{T} -character is well defined as a formal power series.

Since Y is supported on the curves $C_{\alpha\beta}$, we have $\mathcal{I} = \mathcal{O}_X$ on the intersection of three or more U_α . Therefore, only the \mathfrak{C}^0 and \mathfrak{C}^1 terms contribute to the calculation. We find,

$$\begin{aligned} \mathcal{T}_{[\mathcal{I}]} &= \bigoplus_{\alpha} \left(\Gamma(U_\alpha) - \sum_i (-1)^i \Gamma(U_\alpha, \mathcal{E}xt^i(\mathcal{I}, \mathcal{I})) \right) \\ &\quad - \bigoplus_{\alpha, \beta} \left(\Gamma(U_{\alpha\beta}) - \sum_i (-1)^i \Gamma(U_{\alpha\beta}, \mathcal{E}xt^i(\mathcal{I}, \mathcal{I})) \right). \end{aligned} \quad (8)$$

The calculation is reduced to a sum over all the vertices and edges of the Newton polyhedron. In each case, we are given an ideal

$$I = I_\alpha, I_{\alpha\beta} \subset \Gamma(U),$$

and we need to compute

$$\left(\Gamma(U) - \sum_i (-1)^i \text{Ext}^i(I, I) \right)$$

over the ring $\Gamma(U)$, which is isomorphic to $\mathbf{C}[x, y, z]$ in the vertex case and is isomorphic to $\mathbf{C}[x, y, z, z^{-1}]$ in the edge case. We treat each case separately.

4.7 Vertex calculation

Let R be the coordinate ring,

$$R = \mathbf{C}[x_1, x_2, x_3] \cong \Gamma(U_\alpha).$$

As before, we can assume the \mathbf{T} -action on R is the standard action (4). Consider a \mathbf{T} -equivariant graded free resolution of I_α ,

$$0 \rightarrow F_s \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow I_\alpha \rightarrow 0, \quad (9)$$

such as, for example, the Taylor resolution [38]. Each term in (9) has the form

$$F_i = \bigoplus_j R(d_{ij}), \quad d_{ij} \in \mathbb{Z}^3.$$

The Poincare polynomial

$$P_\alpha(t_1, t_2, t_3) = \sum_{i,j} (-1)^i t^{d_{ij}}$$

does not depend on the choice of the resolution (9). In fact, from the resolution (9) we see that the Poincare polynomial P_α is related to the \mathbf{T} -character of R/I_α as follows:

$$\begin{aligned} Q_\alpha(t_1, t_2, t_3) &:= \text{tr}_{R/I_\alpha}(t_1, t_2, t_3) \\ &= \sum_{(k_1, k_2, k_3) \in \pi_\alpha} t_1^{k_1} t_2^{k_2} t_3^{k_3} \\ &= \frac{1 + P_\alpha(t_1, t_2, t_3)}{(1 - t_1)(1 - t_2)(1 - t_3)}, \end{aligned} \quad (10)$$

where trace in the first line denotes the trace of the \mathbf{T} -action on R/I_α .

The virtual representation $\chi(I_\alpha, I_\alpha)$ is given by the following alternating sum

$$\begin{aligned} \chi(I_\alpha, I_\alpha) &= \sum_{i,j,k,l} (-1)^{i+k} \text{Hom}_R(R(d_{ij}), R(d_{kl})) \\ &= \sum_{i,j,k,l} (-1)^{i+k} R(d_{kl} - d_{ij}), \end{aligned}$$

and, therefore,

$$\mathrm{tr}_{\chi(I_\alpha, I_\alpha)}(t_1, t_2, t_3) = \frac{P_\alpha(t_1, t_2, t_3) P_\alpha(t_1^{-1}, t_2^{-1}, t_3^{-1})}{(1-t_1)(1-t_2)(1-t_3)}.$$

We find the character of the \mathbf{T} -action on the α summand of (8) is given by:

$$\frac{1 - P_\alpha(t_1, t_2, t_3) P_\alpha(t_1^{-1}, t_2^{-1}, t_3^{-1})}{(1-t_1)(1-t_2)(1-t_3)}.$$

Using (10), we may express the answer in terms of the generating function Q_α of the partition π_α ,

$$\begin{aligned} \mathrm{tr}_{R-\chi(I_\alpha, I_\alpha)}(t_1, t_2, t_3) \\ = Q_\alpha - \frac{\overline{Q}_\alpha}{t_1 t_2 t_3} + Q_\alpha \overline{Q}_\alpha \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3}, \end{aligned} \quad (11)$$

where

$$\overline{Q}_\alpha(t_1, t_2, t_3) = Q_\alpha(t_1^{-1}, t_2^{-1}, t_3^{-1}).$$

The rational function (11) should be expanded in ascending powers of the t_i 's.

4.8 Edge calculation

We now consider the summand of (8) corresponding to a pair (α, β) . Our calculations will involve modules over the ring

$$R = \Gamma(U_{\alpha\beta}) = \mathbf{C}[x_2, x_3] \otimes_{\mathbf{C}} \mathbf{C}[x_1, x_1^{-1}].$$

The $\mathbf{C}[x_1, x_1^{-1}]$ factor will result only in the overall factor

$$\delta(t_1) = \sum_{k \in \mathbb{Z}} t_1^k,$$

the formal δ -function at $t_1 = 1$, in the \mathbf{T} -character. Let

$$Q_{\alpha\beta}(t_2, t_3) = \sum_{(k_2, k_3) \in \lambda_{\alpha\beta}} t_2^{k_2} t_3^{k_3}$$

be the generating function for the edge partition $\lambda_{\alpha\beta}$. Arguing as in the vertex case, we find

$$\begin{aligned} -\text{tr}_{R-\chi(I_{\alpha\beta}, I_{\alpha\beta})}(t_1, t_2, t_3) \\ = \delta(t_1) \left(-Q_{\alpha\beta} - \frac{\overline{Q}_{\alpha\beta}}{t_2 t_3} + Q_{\alpha\beta} \overline{Q}_{\alpha\beta} \frac{(1-t_2)(1-t_3)}{t_2 t_3} \right). \end{aligned} \quad (12)$$

Note that because of the relations

$$\delta(1/t) = \delta(t) = t\delta(t),$$

the character (12) is invariant under the change of variables (5).

4.9 The equivariant vertex

The formulas (11) and (12) express the Laurent polynomial $\text{tr}_{T_{[\mathbb{Z}]}}(t_1, t_2, t_3)$ as a linear combination of infinite formal power series. Our goal now is to redistribute the terms in these series so that both the vertex and edge contributions are finite.

The edge character (12) can be written as

$$\frac{F_{\alpha\beta}(t_2, t_3)}{1-t_1} + t_1^{-1} \frac{F_{\alpha\beta}(t_2, t_3)}{1-t_1^{-1}}, \quad (13)$$

where

$$F_{\alpha\beta}(t_2, t_3) = -Q_{\alpha\beta} - \frac{\overline{Q}_{\alpha\beta}}{t_2 t_3} + Q_{\alpha\beta} \overline{Q}_{\alpha\beta} \frac{(1-t_2)(1-t_3)}{t_2 t_3}.$$

and the first (resp. second) term in (13) is expanded in ascending (resp. descending) powers of t_1 .

Let us denote the character (11) by F_α and define

$$V_\alpha = F_\alpha + \sum_{i=1}^3 \frac{F_{\alpha\beta_i}(t_{i'}, t_{i''})}{1-t_i},$$

where $C_{\alpha\beta_1}, C_{\alpha\beta_2}, C_{\alpha\beta_3}$ are the three \mathbf{T} -invariant rational curves passing through the point $X_\alpha \in X^\mathbf{T}$, and $\{t_i, t_{i'}, t_{i''}\} = \{t_1, t_2, t_3\}$.

Similarly, we define

$$E_{\alpha\beta} = t_1^{-1} \frac{F_{\alpha\beta}(t_2, t_3)}{1-t_1^{-1}} - \frac{F_{\alpha\beta} \left(t_2 t_1^{-m_{\alpha\beta}}, t_3 t_1^{-m'_{\alpha\beta}} \right)}{1-t_1^{-1}}.$$

The term $E_{\alpha\beta}$ is canonically associated to the edge. Formulas (11) and (12) yield the following result.

Theorem 1. *The \mathbf{T} -character of $\mathcal{T}_{[\mathcal{I}]}$ is given by*

$$\text{tr}_{\mathcal{T}_{[\mathcal{I}]}}(t_1, t_2, t_3) = \sum_{\alpha} V_{\alpha} + \sum_{\alpha\beta} E_{\alpha\beta}. \quad (14)$$

Lemma 9. *Both V_{α} and $E_{\alpha\beta}$ are Laurent polynomials.*

Proof. The numerator of $E_{\alpha\beta}$ vanishes at $t_1 = 1$, whence it is divisible by the denominator. The claim for V_{α} follows from

$$Q_{\alpha} = \frac{Q_{\alpha\beta}}{1 - t_1} + \dots,$$

where the dots stand for terms regular at $t_1 = 1$. \square

From V_{α} , the equivariant localization formula defines a natural 3-parametric family of measures w on 3-dimensional partitions π_{α} . Namely, the measure of π_{α} equals

$$w(\pi_{\alpha}) = \prod_{k \in \mathbb{Z}^3} (s, k)^{-v_k},$$

where $s = (s_1, s_2, s_3)$ are parameters, (\cdot, \cdot) denotes the standard inner product, and v_k is the coefficient of t^k in V_{α} . We call the measure w the *equivariant vertex measure*.

4.10 Local CY and the topological vertex

We now specialize to the local Calabi-Yau geometry discussed in Section 3.

Let S be a nonsingular, toric, Fano surface with canonical bundle K_S . We view the total space of K_S as an open toric Calabi-Yau 3-fold. Let X be the toric compactification defined in Section 3. By definition,

$$Z'_{DT}(S; q)_{\beta} = Z_{DT}(X; q)_{\beta} / Z_{DT}(X; q)_0, \quad (15)$$

for $\beta \in H_2(S, \mathbb{Z})$.

We may compute the right side of (15) by localization. Let

$$D = X \setminus S$$

denote the divisor at infinity. Let $[\mathcal{I}] \in I_n(X, \beta)$ be a \mathbf{T} -fixed ideal sheaf. We have seen the weights of the virtual tangent representation of $[\mathcal{I}]$ are determined by the vertices and edges of the support of Y . Since β is a class on S , the support of Y lies in K_S except for possibly a finite union of zero dimensional subschemes supported on D . Therefore, as a consequence of the virtual localization formula for the Donaldson-Thomas theory of X , we find,

$$Z'_{DT}(S; q)_\beta = \frac{\sum_n q^n \sum_{[\mathcal{I}] \in I_n(K_S, \beta)} \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))}}{\sum_n q^n \sum_{[\mathcal{I}] \in I_n(K_S, 0)} \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))}}. \quad (16)$$

Here, only the ideal sheaves \mathcal{I} for which Y has compact support in K_S are considered. In particular, the local Donaldson-Thomas theory should be viewed as independent of the compactification X .

The open set K_S has a canonical Calabi-Yau 3-form Ω . There is 2-dimensional subtorus,

$$\mathbf{T}_0 \subset \mathbf{T},$$

which preserves Ω . We will evaluate the formula (16) on the subtorus \mathbf{T}_0 .

Let $U_\alpha \subset K_S$ be a chart with coordinates (4). The subgroup \mathbf{T}_0 is defined by

$$t_1 t_2 t_3 = 1.$$

By Serre duality for a compact Calabi-Yau 3-fold, we obtain a canonical isomorphism

$$\text{Ext}^1(\mathcal{I}, \mathcal{I})_0 = \text{Ext}^2(\mathcal{I}, \mathcal{I})_0^*.$$

We will find the \mathbf{T}_0 -representations to be dual in the local Calabi-Yau geometry as well. Formula (16) will be evaluated by canceling the dual weights and counting signs.

The following functional equation for the character (14) expresses Serre duality. On the subtorus $t_1 t_2 t_3 = 1$, the character is odd under the involution $f \mapsto \bar{f}$ defined by

$$(t_1, t_2, t_3) \mapsto (t_1^{-1}, t_2^{-1}, t_3^{-1}).$$

Below we will see, in fact, each term in (14) is an anti-invariant of this transformation.

A crucial technical point is that no term of (14) specializes to 0 weight under the restriction to \mathbf{T}_0 . Since the specializations are all nonzero, the localization formula for \mathbf{T} may be computed after restriction to \mathbf{T}_0 . We leave the straightforward verification to the reader.

We will split the edge contributions of (14) in two pieces

$$\mathsf{E}_{\alpha\beta} = \mathsf{E}_{\alpha\beta}^+ + \mathsf{E}_{\alpha\beta}^-$$

satisfying

$$\overline{\mathsf{E}}_{\alpha\beta}^+ \Big|_{t_1 t_2 t_3 = 1} = -\mathsf{E}_{\alpha\beta}^- \Big|_{t_1 t_2 t_3 = 1}. \quad (17)$$

The total count of (-1) 's that $\mathsf{E}_{\alpha\beta}$ contributes is then determined by the parity of the evaluation of $\mathsf{E}_{\alpha\beta}^+$ at the point $(t_1, t_2, t_3) = (1, 1, 1)$. Concretely, we set

$$F_{\alpha\beta}^+ = -Q_{\alpha\beta} - Q_{\alpha\beta} \overline{Q}_{\alpha\beta} \frac{1 - t_2}{t_2}$$

and define $\mathsf{E}_{\alpha\beta}^+$ in terms of $F_{\alpha\beta}^+$ using the same formulas as before. A straightforward check verifies (17).

Observe that

$$\mathsf{E}_{\alpha\beta}^+ \Big|_{t_1=1} = \left(m_{\alpha\beta} t_2 \frac{\partial}{\partial t_2} + m'_{\alpha\beta} t_3 \frac{\partial}{\partial t_3} - 1 \right) F_{\alpha\beta}^+,$$

Hence, we conclude

$$\mathsf{E}_{\alpha\beta}(1, 1, 1) \equiv f(\alpha, \beta) + m_{\alpha\beta} |\lambda_{\alpha\beta}| \pmod{2}, \quad (18)$$

where the function $f(\alpha, \beta)$ was defined in (6). The second term in (18) comes from applying $\frac{\partial}{\partial t_2}$ to the $(1 - t_2)$ factor in the $Q\overline{Q}$ -term.

Naively, a similar splitting of the vertex term is given by defining F_α^+ to be equal to

$$Q_\alpha - Q_\alpha \overline{Q}_\alpha \frac{(1 - t_1)(1 - t_2)}{t_1 t_2}.$$

However, the definition is not satisfactory since it leads to rational functions and not polynomials: the

$$\frac{(1 - t_1)(1 - t_2)(1 - t_3)}{t_1 t_2 t_3} \quad (19)$$

factor in the $Q\overline{Q}$ -term in (11) can be split in three different ways and no single choice can serve all terms in the $Q\overline{Q}$ -product. The correct choice of the splitting is the following. Define the polynomial Q'_α by the following equality

$$Q_\alpha = Q'_\alpha + \sum_{i=1}^3 \frac{Q_{\alpha\beta_i}}{1 - t_i}.$$

Now for each set of bar-conjugate terms in the expansion of the $Q\bar{Q}$ -product, we pick its own splitting of (19), so that, for example, the term

$$\frac{Q_{\alpha\beta}\bar{Q}_{\alpha\beta}}{(1-t_1)(1-t_1^{-1})} \frac{(1-t_1)(1-t_2)}{t_1 t_2}$$

cancels the corresponding contribution of $F_{\alpha\beta}^+$, and for $i \neq j$ the terms

$$\left(\frac{Q_{\alpha\beta_i}\bar{Q}_{\alpha\beta_j}}{(1-t_i)(1-t_j^{-1})} + \frac{\bar{Q}_{\alpha\beta_i}Q_{\alpha\beta_j}}{(1-t_i^{-1})(1-t_j)} \right) \frac{(1-t_i)(1-t_j)}{t_i t_j}$$

are regular and even at $(1, 1, 1)$.

Using splitting defined above, we easily compute

$$V_\alpha^+(1, 1, 1) \equiv Q'_\alpha(1, 1, 1) \pmod{2}. \quad (20)$$

From the discussion in Section 4.4, we find

$$Q'_\alpha(1, 1, 1) = |\pi_\alpha|. \quad (21)$$

Equations (18) and (21) together with Lemma 5 yield the following result.

Theorem 2. *Let \mathcal{I} be a \mathbf{T} -fixed ideal sheaf in $I_n(K_S, \beta)$,*

$$\frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))} = (-1)^{\chi(\mathcal{O}_Y) + \sum_{\alpha\beta} m_{\alpha\beta} |\lambda_{\alpha\beta}|},$$

where the sum in the exponent is over all edges and

$$O(m_{\alpha\beta}) \oplus O(m'_{\alpha\beta})$$

is the normal bundle to the edge curve $C_{\alpha\beta}$.

The number $1 + m_{\alpha\beta}$ has the same parity as the *framing* of corresponding edge, which is a notion introduced in the context of the topological vertex expansion [2]. Comparing the equivariant localization formula for Donaldson-Thomas theory with the topological vertex expansion, specifically with the melting crystal interpretation of the topological vertex expansion found in [18, 34], we obtain our main result.

Theorem 3. *For the toric local Calabi-Yau geometry, the equality,*

$$Z'_{GW}(S; u, v) = Z'_{DT}(S; -e^{iu}, v)$$

holds.

The above result depends upon the evaluation of the Gromov-Witten theory of toric local Calabi-Yau 3-folds via the topological vertex. The topological vertex has been established in the 1-leg case in [33, 21] and in the 2-leg case in [22]. These evaluations are sufficient for the GW/DT correspondence for nonsingular toric local curves and surfaces.

References

- [1] B. Acharya, M. O'Loughlin, B. Spence, *Higher dimensional analogues of Donaldson-Witten theory*, hep-th/9705138
- [2] M. Aganagic, A. Klemm, M. Marino, C. Vafa, *The topological vertex*, hep-th/0305132.
- [3] L. Baulieu, H. Kanno, and I. Singer, *Special quantum field theories in eight and other dimensions*, hep-th/9704167.
- [4] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), 601–617.
- [5] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45–88.
- [6] J. Bryan and R. Pandharipande, *Curves in Calabi-Yau 3-folds and TQFT*, math.AG/0306316.
- [7] J. Bryan and R. Pandharipande, *Residues in Gromov-Witten and Donaldson-Thomas theory*, in preparation.
- [8] J. Cheah, *On the cohomology of Hilbert schemes of points*, JAG **5** (1996), 479–511.
- [9] D.-E. Diaconescu and B. Florea, *Localization and gluing of topological amplitudes*, hep-th/0309143.

- [10] S. Donaldson and R. Thomas, *Gauge theory in higher dimensions*, in *The geometric universe: science, geometry, and the work of Roger Penrose*, S. Huggett et. al eds., Oxford Univ. Press, 1998.
- [11] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173-199.
- [12] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999), no. 5, 1415–1443.
- [13] R. Gopakumar and C. Vafa, *M-theory and topological strings I*, hep-th/9809187.
- [14] R. Gopakumar and C. Vafa, *M-theory and topological strings II*, hep-th/9812127.
- [15] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.
- [16] J. Harvey and G. Moore, *On the algebras of BPS states*, hep-th/9609017.
- [17] S. Hosono, M.-H. Saito, and A. Takahashi, *Holomorphic anomaly equation and BPS state counting of rational elliptic surface*, Adv. Theor. Math. Phys. **1** (1999), 177-208.
- [18] A. Iqbal, N. Nekrasov, A. Okounkov, and C. Vafa, *Quantum foam and topological strings*, in preparation
- [19] M. Kontsevich, *Homological algebra of mirror symmetry*, math.AG/9411018.
- [20] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, JAMS **11** (1998), 119–174.
- [21] C.-C. Liu, K. Liu, and J. Zhou, *A Proof of a conjecture of Marino-Vafa on Hodge integrals*, math.AG/0308015.
- [22] C.-C. Liu, K. Liu, and J. Zhou, *A formula of 2-partition Hodge integrals*, math.AG/0310272.
- [23] A. Losev, A. Marshakov, N. Nekrasov, *Small instantons, little strings, and free fermions*, hep-th/0302191.

- [24] A. Losev, G. Moore, N. Nekrasov, and S. Shatashvili, *Four-dimensional avatars of two-dimensional RCFT*, hep-th/9509151.
- [25] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory II*, math.AG/0406092.
- [26] D. Maulik and R. Pandharipande, *Foundations of Donaldson-Thomas theory*, in preparation.
- [27] G. Moore, N. Nekrasov, and S. Shatashvili, *Integrating over Higgs branches*, hep-th/9712241,
- [28] G. Moore, N. Nekrasov, and S. Shatashvili, *D-particle bound states and generalized instantons*, hep-th/9803265.
- [29] N. Nekrasov and A. Okounkov, in preparation.
- [30] N. Nekrasov and A. Okounkov, *The equivariant vertex*, in preparation.
- [31] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$ superconformal six dimensional theory*, hep-th/9802068.
- [32] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Birkhauser, 1980.
- [33] A. Okounkov and R. Pandharipande, Hodge integrals and invariants of the unknot, math.AG/0307209.
- [34] A. Okounkov, N. Reshetikhin, and C. Vafa, *Quantum Calabi-Yau and classical crystals*, hep-th/0310061.
- [35] R. Pandharipande, *Three questions in Gromov-Witten theory*, in *Proceedings of the ICM 2002, Vol. II*, Higher Education Press, 2002.
- [36] A. Polyakov, *Gauge fields and strings*, Harwood Academic Publishers, 1987
- [37] R. Stanley, *Enumerative combinatorics*, Cambridge University Press, 1999.
- [38] D. Taylor, *Ideals generated by monomials in an R-sequence*, Ph. D. thesis, University of Chicago, 1966.

- [39] C. Taubes, *Gr = SW: counting curves and connections*, in *Seiberg Witten and Gromov invariants for symplectic 4-manifolds*, R. Wentworth, ed., International Press, 2000.
- [40] R. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3 fibrations*, JDG **54** (2000), 367–438.
- [41] E. Witten, *Chern-Simons gauge theory as a string theory*, hep-th/9207094.
- [42] E. Witten, *BPS bound states of D0-D6 and D0-D8 systems in a B-field*, hep-th/0012054.

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